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INVESTIGATION OF NUMERICAL METHOD FOR SOLUTION OF FAVRE-AVERAGED NAVIER-STOKES EQUATION SYSTEM

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Abstract

Numerical schemes with various approximations of the second derivative in diffusive fluxes are investigated with the aim of finding an optimal one. Calculation results for two model equations are compared with their analytical solutions. Conclusions are made about an optimal scheme on the basis of this comparison.

Introduction

Work, described in this paper, was accomplished on the first stage of the diploma practice. CFD group, where an author has a practical work during the education in MIPhT, comparatively lately has proceeded to consideration of problems with viscosity and turbulence. For this purpose a new numerical method is developed, some good results are already obtained. But before computation of complex 3D flows we need to test thoroughly the method in use. So our group is now engaged in calculation of various test problems, one of which has been proposed to the author.

For calculation of inviscid laminar flows many numerical schemes are developed, they are rather well investigated and optimal class of schemes is founded (Godunov-type schemes using TVD-principle [1-5]). Advantage of these schemes is that they take into account the physical features of the phenomenon. Godunov-type schemes use an exact solution of the Riemann problem about decay of an arbitrary discontinuity. TVD-principle is connected with the 2nd law of thermodynamics.

In case of the Navier-Stokes equations the viscous terms, proportional to the second derivatives of gas parameters, are added to the convective those. The known approximations of these derivatives don't take into account the physical features of the phenomenon. Moreover, for Navier-Stokes equations there is no task with exact solution that could be used for the construction of numerical scheme.

Usual practice is to use for the viscous terms approximation the simplest scheme - central-difference one This scheme has a number of positive features: it has small stencil, consisting of three points; this stencil is symmetric, that correctly reflects the symmetry of disturbances propagation due to diffusion; this scheme provides the second order accuracy in the approximation of the space derivatives (for the uniform grid). But this scheme isn't proved to be the best.

In this work an attempt to find an optimal scheme for approximation of derivatives in diffusive fluxes is presented.

Approach to this problem is based on using simple model equations having analytical solutions. These equations keep the main properties of Navier-Stokes system. Several test problems for these equations are solved numerically. Numerical solution is compared with the exact one.

For viscous terms modeling an approximation depending upon the single parameter is considered. The parameter is varied and function characterizing errors is calculated for test problems. On the basis of this function the conclusions are made about the quality of the scheme with the given parameter value.

1. Viscous Burgers equation and numerical method for its solution

Model equations used in the present paper can be derived from the Navier-Stokes equations upon the circumstance of some simplifying assumptions:

1) Incompressible medium is considered ($\rho = const$);

2) Flow is one-dimensional;

3) Dependence $\nu(T)$ is disregarded, that allows to exclude from consideration the energy equation.

4) The longitudinal pressure gradient is absent $\left(\frac{\partial P}{\partial x} = 0\right)$.

Then the momentum equation is written as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}. \qquad (1)$$

It is the so-called Burgers equation , which is a sufficiently good model, as it has all peculiarities, inherent to the system of Navier-Stokes equations.

At the absence of convective terms Burgers equation reduces to usual diffusion equation:

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2}. \qquad (2)$$

Numerical method

Numerical scheme for the solution of the equations in consideration is obtained by integral method. Then the general form of scheme is:

$$u_j^{n+1} = u_j^n - \frac{\tau}{h}\left[f_{j+1/2} - f_{j-1/2} \right],$$

where the concrete way of the approximation of fluxes $f_{j+1/2}$ and $f_{j-1/2}$ defines the concrete scheme. For the scheme to have a second approximation order in space the function u is considered to be linear within each cell. It is represented as follows:

$$u(x, t_n) = u_j^n + \left(\frac{\partial u}{\partial x}\right)_j^n \left(x - x_j\right), \quad x \in [x_{j-1/2}, x_{j+1/2}]$$

For the approximation of $\left(\frac{\partial u}{\partial x}\right)_j^n$ the Kolgan method [2] is used.

Predictor-corrector procedure is used for providing the second order of approximation in time.

For calculation of convective fluxes $f_{j+1/2} = \frac{u_{j+1/2}^2}{2}$ the Riemann problem about the decay of arbitrary discontinuity between two flows with parameters $u_L = u_j^n + \left(\frac{\partial u}{\partial x}\right)_j^n \frac{h}{2}$ and $u_R = u_{j+1}^n - \left(\frac{\partial u}{\partial x}\right)_{j+1}^n \frac{h}{2}$ is solved exactly on each cell boundary

Described numerical scheme is an explicit monotone scheme of the second approximation order in space and time. It is well-known Godunov-Kolgan-Rodionov scheme[1-4].

Various approximations of the second derivative may be proposed for the simulation of viscous terms $\nu \frac{\partial^2 u}{\partial x^2}$ in equations (1),(2). Due to the symmetry of disturbances propagation by diffusion the stencil must be symmetric. In addition, diffusion equation (2) is linear. This fact allows to hope that optimal approximation may belong to the class of linear approximations. Therefore, let's consider a general form of the second derivative linear approximation for the uniform grid using the five-point stencil:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n = \frac{a_{-2}u_{j-2}^n + a_{-1}u_{j-1}^n + a_0 u_j^n + a_1 u_{j+1}^n + a_2 u_{j+2}^n}{h^2}$$

For the second order of accuracy it is necessary:

$$a_{-1} = a_1, \quad a_{-2} = a_2 = \frac{1 - a_1}{4}, \quad a_0 = \frac{-2(1 + 3a_1)}{4}.$$

This results in the following formula :

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n = a_1 \Delta_1 + (1 - a_1)\,\Delta_2 \qquad (3)$$

where $\Delta_1 = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2}$ is central difference approximation using the close stencil;

$\Delta_2 = \frac{u_{j-2}^n - 2u_j^n + u_{j+2}^n}{(2h)^2}$ is central difference approximation using the distant stencil.

2. Principles of schemes' comparison

Linear schemes for the diffusion equation were investigated by von Neumann spectral method [6], which will be considered below. Using this method the condition of scheme stability $a_1 \geq 0$ and time step limitation is obtained: $\begin{cases} C \leq 2(1 - a_1), & 0 \leq a_1 < 0.5 \\ C \leq \frac{1}{2a_1}, & a_1 \geq 0.5 \end{cases}$, $C = \nu \frac{\tau}{h^2}$.

In connection with these conditions, only schemes with parameter $a_1 \in [0, 3]$ are compared, since for $a_1 > 3$ time step limitation becomes too strict and there is no sense to use such schemes.

In addition, the main purpose of this work is the choice of an optimal scheme for solving stationary problems. In this class of problems local time stepping is usually accepted. In this method time advance realizes in each cell with the maximum possible time step permitted by the stability condition $\tau_{max} = \min\left[\frac{h^2}{\nu} C_{max}, \frac{h}{|u|}\right]$. While being very convenient and quick procedure the calculation with the local time step has nothing in common with the realistic flow evolution, which sometimes lead to the breakdown of solution. To avoid this it is necessary to multiple τ_{max} by some stability coefficient K <1, depending on the properties of calculated flow but not on the scheme parameter u_{j+1}^n. That's why schemes with the same K are compared, $\tau = \tau_{max} \cdot K$, $K \leq 1$. Parameter a_1 is varied from 0 to 3 with the step 0.05. Calculations are performed for three stability coefficients K = 1, 0.5 and 0.1. For each calculation function of errors $F = \sum_{j=1}^{n} \left| u_j - u_j^{ex} \right|$ (u_j - numerical u_j^{ex} - exact solution of the given problem) is computed. On the basis of this function the scheme comparison is fulfilled.

Results of scheme comparison on the basis of error function for all test tasks are represented at Fig.2-6. Value of the error function F is plotted along the vertical axis, value of the parameter a_1 is plotted along the horizontal axis. Curves are plotted for all three values of *K*.

3. Test calculations for the diffusion equation

Let's consider two Cauchy problems for the diffusion equation (2).

1) <u>The first test task</u>: an action of diffusion on δ - function. In this case in the initial time moment

$$u(x,0) = \delta\,(x - x_0)\,.$$

The solution of this task is the Gauss function [6]:

$$u(x,t) = \frac{1}{\sqrt{4\pi\,\nu(t-t_0)}}\exp\{-\frac{(x-x_0)^2}{4\,\nu(t-t_0)}\}$$

Calculations show that when $a_1 < 1$ scheme is nonmonotone (see the velocity profile *u(x)* at Fig. 1)

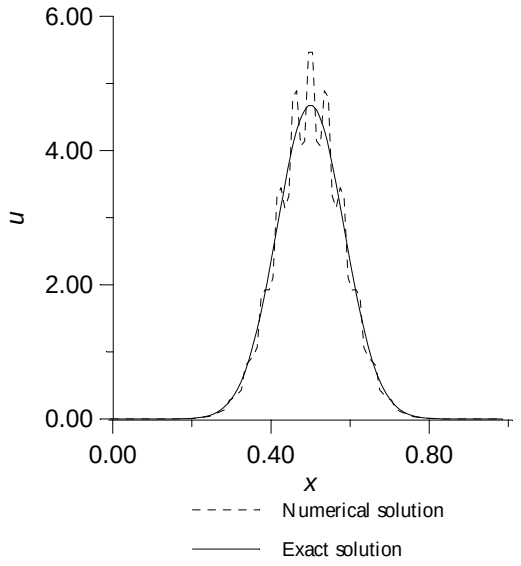


Fig.1 The first test task for the diffusion equation. Velocity profile for. a_1**<1**

Function of errors $F(a_1)$ for this computation is represented at Fig.2,a. When K = 1 an optimal value of a_1 is $a_1 \approx 2$. As $K \to 0$, an optimum displaces toward $a_1 \approx 4/3$. The thing is that the scheme with $a_1 \approx 4/3$ has the third approximation order in space. Using finite time steps ($K \approx 1$) we obtain the scheme with mixed approximation order (the third in space and the second in time). But as K tends to zero the role of time terms becomes less sufficient and order of x approximation begins to play the cardinal role. This explains the good quality of this scheme for small time steps.

It is interesting that with the diminishing of K for $a_1 = const$, the errors diminish too, if $a_1 < 1.6$, and rise, if $a_1 > 2$. This fact can be explained as follows. From the formula (3) it can be seen that when $a_1 \approx 1$ only closest cells j-1, j and j+1 play the principal role in the approximation of $\left(\frac{\partial^2 u}{\partial x^2}\right)^n_j$. It is correct only for small time steps when disturbance from more distant cells doesn't have sufficient time to rich the cell j and make a contribution in averaged flux (during the time τ viscous disturbances propagate at the distance $\Delta \sim \sqrt{\nu\,\tau}$). So, the schemes with $a_1 \approx 1$ reveal the better quality when K is small. With the increase in a_1 the role of more distant cells j-2 and j+2 in the second derivative approximation increases and the better scheme quality is achieved for the larger value of K.

Calculations of the same task for two times less finishing time were also implemented, results are represented in Fig.1,b. It can be seen that the qualitative character of the curves including the positions of the error function minimum is the same as in the first case (compare Fig.2,a and Fig.2,b).

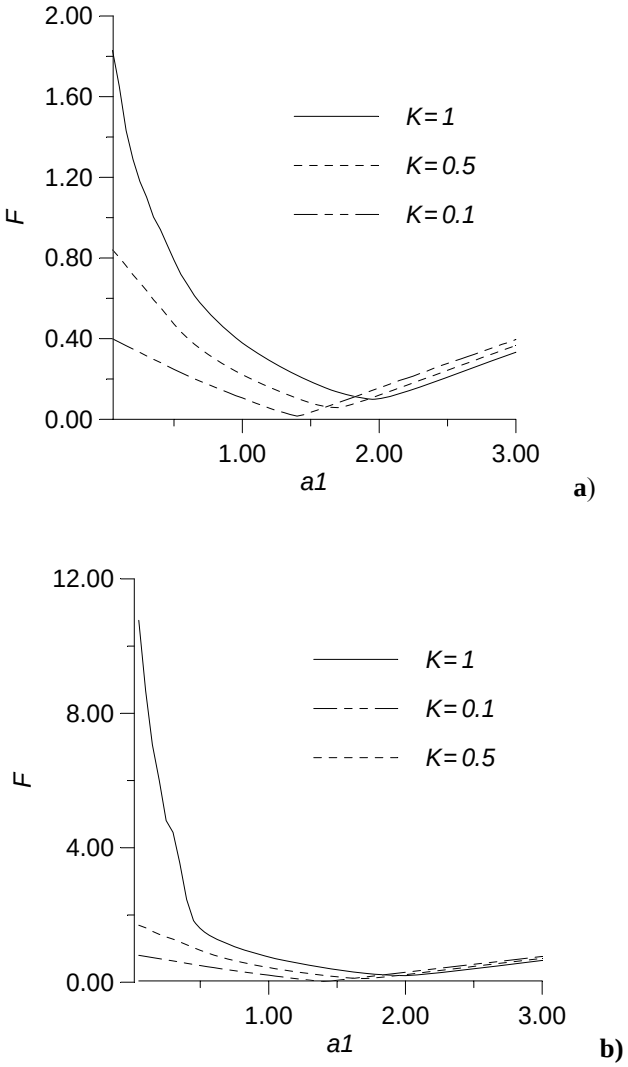


Fig. 2. The first test task for the diffusion equation. Plots of the error function:

a) computation up to the time T_0 **;**

b) computation up to the time $T_0/2$

2) <u>The second test task</u> is determined by the following initial conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nu\dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = \displaystyle\sum_{k=1}^{6}\sin(\frac{\pi x}{k\,h}) \end{cases}.$$

This task is considered with the aim of tracing the evolution of concrete set of harmonics with known wave

numbers. An exact solution of this problem is:

$$u(x,t)=\sum_{k=1}^{6}\exp\{-\left(\frac{\pi}{k\,h}\right)^2 \nu t\}\sin\frac{\pi x}{k\,h}$$

Qualitative character of the result (Fig.3) is in agreement with that for the first task, though $a_{1\,opt}$ are moved to the right (compare Fig.3 and Fig.2).

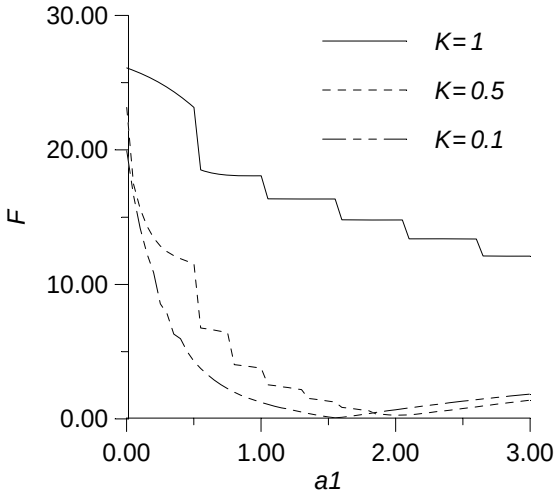


Fig. 3. The second test task for the diffusion equation. Plot of the error function

4. Explanation of the results obtained for the diffusion equation

For the explanation of the results obtained for the diffusion equation an analysis of difference schemes with the von Neumann spectral method is used. The idea of this method consists in the expansion of exact and numerical solutions in Fourier series on the layer t =*const* :

$$u=\sum U_k(t)\exp\left\{i\frac{2\pi kx}{L}\right\}=\sum U_k(t)\exp\{i\varphi j\},$$

where $\varphi=\frac{2\pi k h}{L}$, $\quad j=\frac{x}{h}$.

For an exact solution of diffusion equation we have:

$$\frac{\partial U_k}{\partial t}=-\frac{\nu\varphi^2}{h^2}U_k,$$

from which

$$U_k(t)=U_{k0}\exp\left\{-\frac{\varphi^2\nu}{h^2}t\right\}.$$

Factor of transition from n to (n+1) time layer:

$$\lambda_{ex}=\frac{U_k(t+\tau)}{U_k(t)}=\exp\{-C\varphi^2\}.$$

For the numerical solution we can represent

$$\lambda=\alpha_0+2\sum\alpha_k\cos k\varphi,\ \text{where}$$

$$\alpha_{-4}=\alpha_4=\frac{C^2}{32}(1-a_1)^2$$

$$\alpha_{-3}=\alpha_3=\frac{C^2}{4}a_1(1-a_1)$$

$$\alpha_{-2}=\alpha_2=\frac{C}{4}\left[(1-\frac{C}{2})-(1+C)\,a_1+\frac{7}{2}Ca_1^2\right]$$

$$\alpha_{-1}=\alpha_1=\frac{C}{4}a_1\left[(4-C)-7Ca_1\right]$$

$$\alpha_0=\left[1+\frac{C}{4}\left(\frac{3C}{4}-2\right)\right]+\frac{C}{4}\left(\frac{5C}{2}-6\right)a_1+\frac{35C^2}{16}a_1^2$$

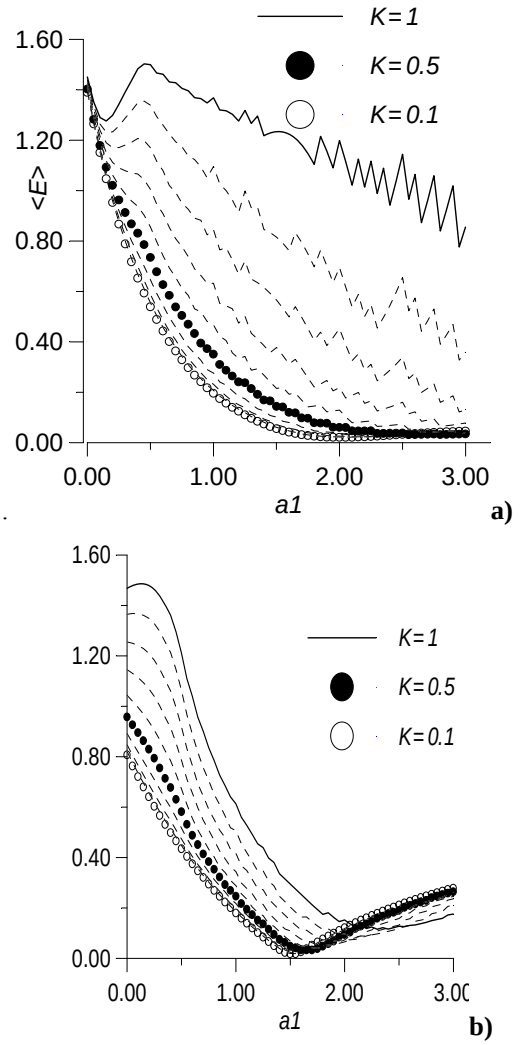


Fig. 4. Results of von Neumann spectral method using. Plots of error function:
a) Averaging on all resolved wavelengths;
b) Smaller interval of averaging

If $\left(\frac{\partial^2 u}{\partial x^2}\right)^n_j$ is approximated with the second order of accuracy, the phase errors are absent. Amplitude error during one time step may be calculated as λ/λ_{ex}. Function, characterising errors accumulated during the same

time $E(\varphi) = \left| \left(\lambda/\lambda_{ex}\right)^{0.5/C} - 1 \right|$ is used . for the investigation of schemes by the von Neumann method. In the common case phase φ belongs to the interval $[-\pi, \pi]$; phase $\varphi = \pm\pi$ corresponds to the wavelength *2 h* . Let's consider the value of amplitude errors averaged over all wavelengths resolved by the scheme - $\langle E \rangle = \frac{1}{\pi} \int_0^{\pi} E(\varphi)\, d\varphi$. It can be chosen as the index of scheme quality. The result of such averaging is shown in Fig.4,a. It is seen that the dependence of error function $\langle E \rangle$ on a_1 isn't similar to the dependence F (a_1) for the first test task (smoothing of delta function), but there are some analogies with the results for the second test task (compare Fig.3 and 4,a). It is reasoned by the fact that in spectral expansion of the solution for the latter task harmonics with small wavelengths $\approx 2h$ play an essential role, while in the former task these harmonics are slightly represented, since the shape of solution is smooth and fluent. Therefore let's take a smaller interval of averaging: $\langle E \rangle = \frac{3}{2\pi} \int_0^{\frac{2\pi}{3}} E(\varphi)\, d\varphi$. Then the dependence $\langle E \rangle$ on a_1 (Fig.4,b) is in good agreement with error function F (a_1) for the Gauss function including the positions of $a_{1\,opt}$ (compare Fig.1 and Fig.4,b)

So, the results of test tasks are in accordance with the theory. Therefore scheme comparison on the basis of error function F is valid and may be used for the choice of an optimal scheme.

6. Test calculations for the Burgers equation

Von Neumann method doesn't operate in case of nonlinear Buregers equation (1). That's why this is the case where practically single way to compare schemes quality is analysis of test tasks solution using these schemes.

The problem of internal structure of the shock wave front within the framework of continuous medium model is chosen as the test task. Boundary conditions are the following: $\begin{cases} u(-\infty) = u_L \\ u(+\infty) = u_R \end{cases}$ ($u_L > u_R$ for the existence of the solution in the form of the shock wave.)

This task has analytical solution:

$$u(x,t) = D - \frac{u_L - u_R}{2} \operatorname{th} \frac{u_L - u_R}{4\nu}(x - D\,t)$$

where $D = \frac{u_L + u_R}{2}$ - is shock velocity.

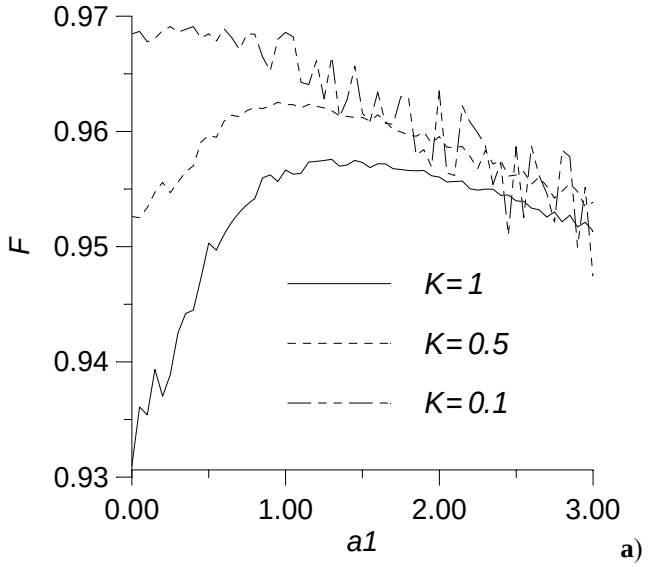
7. Description of the results

1) Nonstationary shock waves (*D*=0.5 and *D*=-0.5)

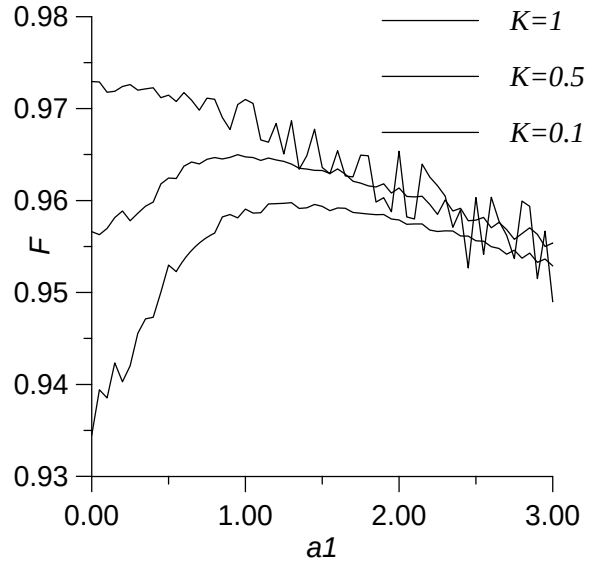
Results of calculating nonstationary shock waves are shown in Fig.5. It is seen that the quality of the schemes is independent from the direction of the wave propagation. It can be explained by the fact that scheme for convective terms takes into account the direction of flow propagation (it's the well-known upwind principle), and scheme for diffusive terms is symmetric

2) Stationary shock wave(D=0)

Results for the stationary shock wave are represented in Fig.6. It's easy to see that in this case the scheme quality doesn't depend on *K*. If we refer to the results obtained in nonstationary case we can notice that those for $K \to 0$ qualitatively agree with the results for stationary problem. It is clear, since when $K \to 0$ the role of time terms is insignificant even for nonstationary problems, i.e. even nonstationary process is described as quasi-stationary



a)



b)

Fig. 5. Plots of error function for the Burgers equation:
a) Shock wave propagating to the right (*D*=0.5);
b) Shock wave propagating to the left (*D*=-0.5)

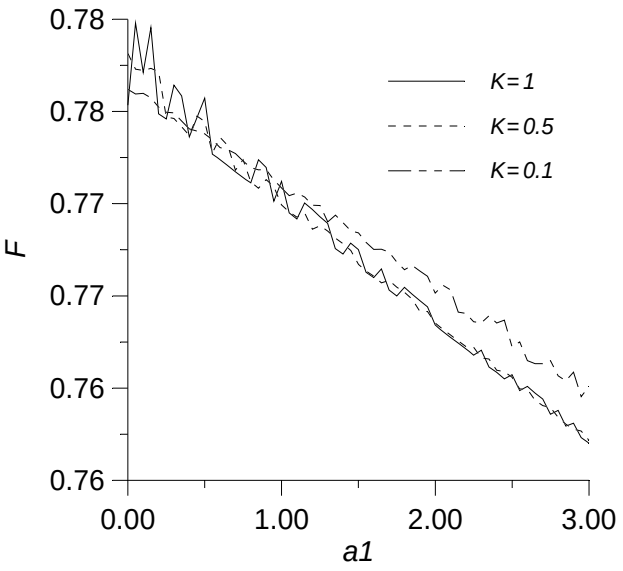


Fig. 6. Plot of error function for the Burgers equation. Stationary shock wave (*D* =0)

Two requirements are imposed to the numerical scheme if the stationary task is solved using time-marching approach with local time stepping:

1) An optimal stationary solution must be obtained. It doesn't change in marching from one stationary layer to another. Therefore, we can conclude about the scheme quality for stationary solutions when calculating nonstationary problem with $K \to 0$.

2) Nonstationary process of approaching to stationary solution is to be described as precise as possible (since the higher quality of describing nonstationary process, the less disturbances are introduced by the scheme in this process and the faster the solution converges). That's why we are also interested in the scheme quality when calculating nonstationary flows (with $K \sim 1$).

So, the choice of an optimal scheme depends upon the purpose we follow. Let's turn to the optimal scheme for the diffusion equation. If it is required to obtain stationary flow as fast as possible, i.e. when it is necessary to describe nonstationary process ($K \sim 1$) with maximum accuracy, schemes with $a_1 \approx 2$ can be recommended for using. But if the accuracy of the stationary solution is more important, it is better to use $a_1 \approx 4/3$. In principle, an adaptive algorithm is possible, when a_1 gradually decreases from 2 to 4/3 with decrease in convergence parameter $\max_j \frac{u_j^{n+1} - u_j^n}{u_j^n}$.

It must be noted, however, that in the case of the Burgers equation dependence $F(a_1)$ is absolutely another than that for the diffusion equation. It appears to be explained by the non-linear interactions of second derivative approximation errors with convective terms and with errors of their approximation. But it may be noted that the relative change of errors from scheme to scheme in the case of the Burgers equation is very small (about 2-3%). It is far less than the relative change for the diffusion equation. (about 100%). And so there is a hope that one can use the scheme with $a_1 = 1$ without considerable risk to loose in quality. This scheme has a three-point stencil and therefore requires minimum quantity of calculations

Conclusions

1) For the diffusion equation schemes with $a_1 < 1$ are not monotone. When $K \sim 1$ an optimal scheme for this equation is the scheme with $a_1 \sim 2$; when $K \to 0$ an optimal scheme for it is one with $a_1 \sim 4/3$. It's possible to use an adaptive algorithm of changing a_1 from 2 to 4/3 as solution converges (i.e. the parameter of convergence ($\max_j \frac{u_j^{n+1} - u_j^n}{u_j^n}$) diminishes).

2) For choosing an optimal approximation of the viscous terms in the Navier-Stokes equations it is necessary to take into account their non-linear interaction with the convective terms. But tests show that in problems of such class, where both convection and diffusion are present, errors slightly depend upon the choice of second derivative approximation (if the second order of accuracy is guaranteed). That's why scheme of central differences ($a_1 = 1$), which requires minimal volume of calculations, can be recommended for these tasks.

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