Towards a stable and accurate coupling of compressible and incompressible flow solvers

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1. Motivation and objectives

A consistent and accurate scheme does not guarantee an accurate solution of the Partial Differential Equation (PDE). According to the famous paper by Lax *et al.* (1956), a consistent approximation of a well-posed linear PDE will converge to the true solution if and only if the scheme is stable. This theorem is of great importance, since it is in general easier to prove stability than to prove convergence directly.

Lax and Richtmyer's theorem implies that convergence to the true solution at a fixed time T is achieved as the grid size $h \to 0$. With this definition of stability, the error of the solution may, in fact, grow exponentially in time for realistic mesh sizes. As pointed out by Carpenter *et al.* (1993), it is therefore important to devise schemes that do not allow a growth in time that is not called for by the differential equation. Such schemes are called strictly (or time) stable.

One way to obtain a strictly stable and accurate scheme for a linear problem is to i) approximate the derivatives of the initial boundary value problem with accurate, non dissipative operators that satisfy a summation by parts (SBP) formula (Kreiss *et al.* (1974)), and ii) use a specific procedure, referred to as the Simultaneous Approximation Term (SAT) method (Carpenter *et al.* (1994)) for implementation of boundary conditions.

For nonlinear problems, stability by energy estimates can not be obtained in general. However, Strang (1964) showed that if the solution of the nonlinear differential equation is sufficiently smooth, convergence follows if the linearized approximation is stable, meaning that linear stability is a necessary and sometimes sufficient condition for stability. Nonlinear instabilities (due to under-resolved features) are usually divided into two different phenomena: shocks and aliasing. These factors can make central difference schemes unstable when applied to nonlinear problems on under-resolved grids. In this case, some type of artificial damping is needed, such as filtering or artificial dissipation.

We begin by discussing the SBP property for the first and second derivative difference operator and then introduce the SAT method. Then, we will show how to combine two different compressible SBP schemes by using the SAT method. Finally, we will use the same methodology to combine a compressible fourth order accurate SBP discretization with an incompressible unstructured SBP discretization of CDP.

1.1. *The SBP property*

SBP operators arise naturally in various nodal based space discretizations. The main classes are spectral methods, finite elements, finite differences and finite volumes. High order accurate SBP operators for finite differences were derived in Kreiss *et al.* (1974) and in Mattsson *et al.* (2004). The SBP property for the unstructured node centered finite volume method was shown in Nordström *et al.* (2003) Recently, we developed an unstructured SBP discretization of CDP. To simplify the introduction of the SBP and SAT concepts, we will restrict the rigorous analysis to one dimensional problems. The generalization to multi dimensional problems is straightforward but somewhat tedious and lengthy to describe in detail.

Before we start describing the SBP property, some definitions are needed. Let the inner product for real valued $m \times 1$ vector functions $u, v \in L^2[l, r]$ be defined by $(u, v) = \int_l^r u^T v\, dx$, and let the corresponding norm be $\|u\|^2 = (u, u)$. The domain $(l \leq x \leq r)$ is discretized using *N+1* grid points.

The numerical approximation of the $k : th$ component of u at grid point x_j is denoted v_j^k. We define a discrete solution vector $v^T = [v_0^T, v_1^T, \cdots, v_N^T]$, where $v_0 = [v_0^1, v_0^2, \cdots, v_0^m]^T$ is the discrete approximation of u at the left boundary. We define an inner product for discrete real valued vector-functions $u, v \in \mathbf{R^{N+1\times m}}$ by $(u, v)_H = u^T H v$, where $H = H^T > 0$, with a corresponding norm $\|v\|_H^2 = v^T H v$. The vectors

$$\hat{e_0} = [1, 0, ..., 0]^T , \quad \hat{e_N} = [0, ..., 0, 1]^T , \tag{1.1}$$

of size $N + 1 \times 1$, will frequently be used in subsequent sections.

1.1.1. *The first derivative*

Consider the hyperbolic scalar equation, $u_t + u_x = 0$ (excluding the boundary condition). Notice first that $(u, u_t) + (u_t, u) = d/dt\|u\|^2$. Integration by parts leads to

$$\frac{d}{dt}\|u\|^2 = -(u, u_x) - (u_x, u) = -u^2|_l^r , \tag{1.2}$$

where we introduce the notation $u^2|_l^r \equiv u^2(r, t) - u^2(l, t)$. To simplify the notation for the continuous problem, we will denote $u(k, t)$ by u_k. A discrete approximation can be written $v_t + D\, v = 0$. We introduce the following definition:

Definition 1.1. *A difference operator $D = H^{-1}Q$ approximating $\partial/\partial\, x$ is said to be a first derivative SBP operator if i) $H = H^T > 0$ and ii) $Q + Q^T = B = diag\,(-1, 0 \ldots, 0, 1)$.*

By multiplying the semi discrete approximation by $v^T H$ and by adding the transpose and utilizing Definition 1.1, we obtain

$$v^T H\, v_t + v_t^T H^T\, v_t = \frac{d}{dt}\|v\|_H^2 = -v^T (Q + Q^T) v = v_0^2 - v_N^2 . \tag{1.3}$$

(1.3) is a discrete analog to the integration by parts (IBP) formula (1.2) in the continuous case.

1.1.2. *The second derivative*

For parabolic problems, we need an SBP operator also for the second derivative. Consider the heat equation $u_t = u_{xx}$. Multiplying by u and integration by parts leads to

$$\frac{d}{dt}\|u\|^2 = (u, u_{xx}) + (u_{xx}, u) = 2uu_x|_a^b - 2\|u_x\|^2 . \tag{1.4}$$

A discrete approximation is given by $v_t = D_2 v$. We introduce the following definition

Definition 1.2. *A difference operator $D_2 = H^{-1}(-M + BS)$ approximating $\partial^2/\partial\, x^2$ is said to be a second derivative SBP operator if $x^T (M + M^T) x \geq 0$, if S includes an approximation of the first derivative operator at the boundary and $B = diag\,(-1, 0 \ldots, 0, 1)$.*

By multiplying the semi discrete approximation by $v^T H$, adding the transpose, and utilizing Definition 1.2, we obtain

$$\frac{d}{dt}\|v\|_H^2 = 2v_N (Sv)_N - 2v_0 (Sv)_0 - v^T (M + M^T) v . \tag{1.5}$$

To obtain an energy estimate, it suffices that $x^T(M + M^T)\, x \geq 0$, assuming that the boundary terms are correctly implemented.

To obtain stability estimates for a mixed hyperbolic-parabolic problem it is necessary that the norm H used in the construction of the first and second derivative SBP operators are the same. High order accurate second derivative SBP operators were developed in Mattsson *et al.* (2004). For completion, the 4th order accurate operators based on diagonal norms, for both the first and second derivative approximations, are listed in Appendix 5.

1.2. *The SAT method*

By using an SBP operator, a strict stable approximation for a Cauchy problem is obtained. Nevertheless, because the SBP property alone does not guarantee strict stability for an initial boundary value problem, a specific boundary treatment is also required. Imposition of the boundary condition explicitly, i.e. combination of the difference operator and the boundary operator into a modified operator, usually destroys the SBP property. In general, this makes it impossible to obtain an energy estimate. This boundary procedure, often used in practical calculations, is referred to as the injection method and can result in an unwanted exponential growth of the solution; see for example Strand (1996) and Mattsson (2003).

The basic idea behind the SAT method is to impose the boundary conditions weakly as a penalty term, such that the SBP property is preserved and such that we get an energy estimate. As an example of the simple yet powerful SAT boundary procedure, we consider the hyperbolic scalar equation,

$$u_t + u_x = 0, \quad l \leq x \leq r, \quad t \geq 0, \quad u_l = g_l \,. \tag{1.6}$$

Integration by parts leads to

$$\frac{d}{dt}\|u\|^2 = g_l^2 - u_r^2 \,. \tag{1.7}$$

A discrete approximation of (1.6) using an SBP operator to discretized the domain combined with the SAT method for the boundary condition is given by

$$Hv_t + Qv = \tau \hat{e}_0 \left\{v_0 - g_l\right\} \,, \tag{1.8}$$

where $\hat{e}_0$ is defined in (1.1).

The energy method in (1.8) leads to

$$\frac{d}{dt}\|v\|_H^2 = \frac{\tau^2}{2\,\tau - 1} g_l^2 - v_N^2 - (2\tau - 1)\left(v_0 - \frac{\tau}{2\,\tau - 1} g_l\right)^2 \,.$$

Clearly, an energy estimate exists for $\tau > 1/2$. By choosing $\tau = 1$, the energy method leads to

$$\frac{d}{dt}\|v\|_H^2 = g_l^2 - v_N^2 - (v_0 - g_l)^2 \,. \tag{1.9}$$

(1.9) is a discrete analog of the integration by parts formula (1.7) in the continuous case, where the extra term $(v_0 - g_l)^2$ introduces a small additional damping. Note that no artificial dissipation is included.

2. Interface conditions

Consider the following linearized system of equations with an interface at $x = i$

$$\begin{aligned} u_t + Au_x &= (B_l u_x)_x + F_l \quad l \leq x \leq i, \quad t \geq 0 \,, \\ p_t + Ap_x &= (B_r p_x)_x + F_r \quad i \leq x \leq r, \quad t \geq 0 \,, \end{aligned} \tag{2.1}$$

where $F_{l,\,r}$ are forcing functions and A, B_l, B_r are symmetric matrices. Since A is symmetric,

$$A = X \Lambda X^T \,, \tag{2.2}$$

where the columns of X are the eigenvectors to A, and Λ is the eigenvalue matrix.

The energy method applied to (2.1) leads to

$$\frac{d}{dt}(\|u\|_l^2 + \|p\|_r^2) = BT_l + BT_r + DI + FO + IT_i \,, \tag{2.3}$$

where $BT_{l,\,r}$ denotes the boundary terms, FO the forcing terms, DI corresponds to the physical dissipation and IT the interface coupling term. The interface conditions at the interface $x = i$ are given by

$$u = p \;, \; B_l u_x = B_r p_x \;. \tag{2.4}$$

The main focus here is on obtaining a stable and accurate interface coupling for the corresponding semi discrete problem. We can have different forcing $F_l \neq F_r$ and different physical viscosity $B_l \neq B_r$ in the two regions, and the following analysis still holds.

The semi discrete finite difference approximation to (2.1) and (2.4) can be written

$$\begin{aligned} v_t + (D_l \otimes A)v &= ((D_2)_l \otimes B_l)v + P_l^I + P_l^V + P_{li}^I + P_{li}^V + F_l \,, \\ w_t + (D_r \otimes A)p &= ((D_2)_r \otimes B_r)w + P_r^I + P_r^V + P_{ri}^I + P_{ri}^V + F_r \,, \end{aligned} \tag{2.5}$$

where the inviscid interface penalties are given by

$$\begin{aligned} P_{li}^I &= (H_l^{-1} \otimes \Pi_l) e_N \otimes A_-(v_N - w_0) \,, \\ P_{ri}^I &= -(H_r^{-1} \otimes \Pi_r) e_0 \otimes A_+(w_0 - v_N) \,, \end{aligned} \tag{2.6}$$

where

$$A_{+,\,-} = X \Lambda_{+,\,-} X^-$$

and

$$2\,\Lambda_+ = \Lambda + |\Lambda| \;;\; 2\,\Lambda_- = \Lambda - |\Lambda| \;.$$

Here we have introduced the Kronecker product

$$C \otimes D = \begin{bmatrix} c_{0,0}\,D & \cdots & c_{0,q-1}\,D \\ \vdots & & \vdots \\ c_{p-1,0}\,D & \cdots & c_{p-1,q-1}\,D \end{bmatrix} ,$$

where C is a $p \times q$ matrix and D is a $m \times n$ matrix. Two useful rules for the Kronecker product are $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ and $(A \otimes B)^T = A^T \otimes B^T$.

The corresponding viscid interface penalties are given by

$$\begin{aligned}
P_{li}^V &= +(H_l^{-1}\otimes\Psi_l)e_N\otimes(B_l(S_l v)_N - B_r(S_r w)_0) \\
&\quad + (H_l^{-1}\otimes\Phi_l)(S_l^T\otimes B_l)e_N\otimes(v_N - w_0) \\
P_{ri}^V &= -(H_r^{-1}\otimes\Psi_r)e_0\otimes(B_r(S_r w)_0 - B_l(S_l v)_N) \\
&\quad - (H_r^{-1}\otimes\Phi_r)(S_r^T\otimes B_r)e_0\otimes(w_0 - v_N)
\end{aligned} \quad . \tag{2.7}$$

We can have different SBP discretizations in the left (l) and right (r) domain respectively, and the analysis still holds. The energy method applied to (2.5) leads to

$$\frac{d}{dt}(\|v\|_{H_l}^2 + \|w\|_{H_r}^2) = IT^V + IT^I + BT + DI + FO\ . \tag{2.8}$$

BT corresponds to the boundary terms, DI to the physical dissipation and $IT^I = y_l^T M_l\, y_l + y_r^T M_r\, y_r$ to the inviscid part of interface coupling, where

$$M_r = -\begin{bmatrix} I & -\Pi_r \\ -\Pi_r & 2\,\Pi_r - I \end{bmatrix}\ ,\ M_l = -\begin{bmatrix} 2\,\Pi_l - I & -\Pi_l \\ -\Pi_l & I \end{bmatrix}\ ,$$

and

$$y_r = \begin{bmatrix} |\Lambda_+|^{1/2}Xv \\ |\Lambda_+|^{1/2}Xw \end{bmatrix}\ , y_l = \begin{bmatrix} |\Lambda_-|^{1/2}Xv \\ |\Lambda_-|^{1/2}Xw \end{bmatrix}\ .$$

The viscid part of the interface coupling is given by

$$\begin{aligned}
IT^V &= 2v_N^T B_l(Sv)_N(I + \Psi_l + \Phi_l) + 2w_N^T B_l(Sw)_0(-I + \Psi_r + \Phi_r) \\
&\quad - 2v_N^T B_r(Sw)_0(\Psi_l + \Phi_r) - 2w_0^T B_l(Sv)_N(\Psi_r + \Phi_l)
\end{aligned}\ .$$

A stable coupling requires that i) $IT^I \leq 0$, and ii) $IT^V = 0$. The first condition holds if

$$\Pi_l = I\ ,\ \Pi_r = I\ . \tag{2.9}$$

The second condition holds if

$$\Psi_r = \Psi,\ \Psi_l = \Psi - 1,\ \Phi_l = -\Psi,\ \Phi_r = I - \Psi\ . \tag{2.10}$$

Notice that Ψ is a free parameter, but its value will affect the eigenvalues i.e. the stiffness of the problem. To minimize stiffness we chose $\Psi = \frac{I}{2}$. Together, the interface conditions (2.9) and (2.10) lead to a stable interface coupling

$$IT^V + IT^I = +(v_N - w_0)^T A_-(v_N - w_0) - (v_N - w_0)^T A_+(v_N - w_0)\,.$$

3. Computations and further analysis

Here, we want to test and validate the boundary and interface treatment by convecting a Taylor vortex (G.I. Taylor (1918)) across the interface, thereby coupling two domains in two dimensions. The Taylor vortex is an analytic solution to the Navier Stokes equations and has the following form:

$$v_\phi = \frac{\mathcal{M}r}{16\pi\nu^2 t^2}\exp(\frac{-r^2}{4\nu t});\ v_r = v_z = 0,$$

where v_ϕ is tangential velocity, v_r is radial velocity, v_z is spanwise velocity, and r is the distance from the center of the vortex. This circular vortex is convected with the free-stream from one computational domain to another. The size of the domains is approximately 18×15 in terms of the radius of the vortex with 144×96 computational grid points in each domain.

3.1. *Compressible-Compressible coupling*

We compare two different finite difference approximations for the compressible Navier-Stokes equations in two dimensions. The spatial derivatives in both methods are approximated with fourth-order accurate central-differences in the interior. The first type of operators introduced in Xiong (2004) has a non-SBP boundary closure, while the others (see Appendix 5) are constructed with an SBP boundary closure. We use implicit time-advancement with approximate factorization. The two dimensional Navier-Stokes equations are written in primitive variables:

$$u_t + A_1 u_x + A_2 u_y = (B_{11} u_x + B_{12} u_y)_x + (B_{21} u_x + B_{22} u_y)_y + F ,$$

where

$$u^T = [\rho,\, u,\, v,\, T] .$$

Details of the numerical implementation of the compressible code can be found in Xiong (2004). The computational domain consists of two blocks that are patched together to a global domain by using the SAT method by penalizing the characteristic variables, as described in (2.5). We also include the case in which we instead penalize the variables, i.e. we replace the characteristic interface penalties (2.6) with

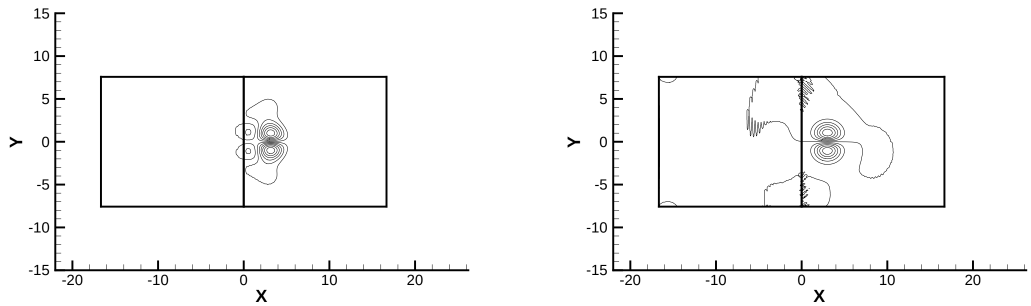
$$\begin{aligned} \tilde{P}^I_{li} &= (H_l^{-1} \otimes \Pi_l) e_N \otimes A(v_N - w_0) \\ \tilde{P}^I_{ri} &= -(H_r^{-1} \otimes \Pi_r) e_0 \otimes A(w_0 - v_N) . \end{aligned} \tag{3.1}$$

Four numerical test cases, referred to as cases A, B, C and D, are analyzed and representing different boundary closures for the interface coupling. For methods A and B we are using the SBP operators, and for methods C and D we are using the non-SBP operators. In addition, methods B and D are implemented using the characteristic penalties, as given by (2.5). For methods A and C we are using the non-characteristic penalties (3.1) to couple the two domains.

The L2 error was computed for the streamwise velocity component u for rather low Reynolds number $Re = 330$ with respect to the radius of the vortex. We observed that methods A and C, corresponding to the non-characteristic penalization, are unstable for the given problem.

To further test methods B and D, viscosity was reduced to zero, thereby establishing a very rigorous test for stability. No artificial dissipation was added. Here it was observed that method D, which is the combination of characteristics penalties with non-SBP boundary operators, also proves to be unstable. This is no surprise, as method D does not lead to an energy estimate, and hence, the technique is not guaranteed to be stable.

To illustrate the nature of instability of the penalty method for non-characteristic penalization and for non-SBP operators, we plot the contours of streamwise velocity when the vortex just crosses the interface for methods A (SBP and non-characteristic penalty) and D (characteristic penalty and non-SBP), for $Re = \infty$ in Fig 1. One can see that the nature of instability is completely different for these two methods. For method A, which is a combination of SBP and non-characteristic penalization, the vortex itself is distorted, due to the violation of characteristic information propagation. However, for method D, which is a combination of characteristic penalty and non-SBP, the vortex



(a) Method A.

(b) Method D.

FIGURE 1. Contours of streamwise velocity when the vortex just crossed the interface.

itself seems to be perfectly transformed, but the numerical errors are accumulated at the interface due to lack of boundary cancellation.

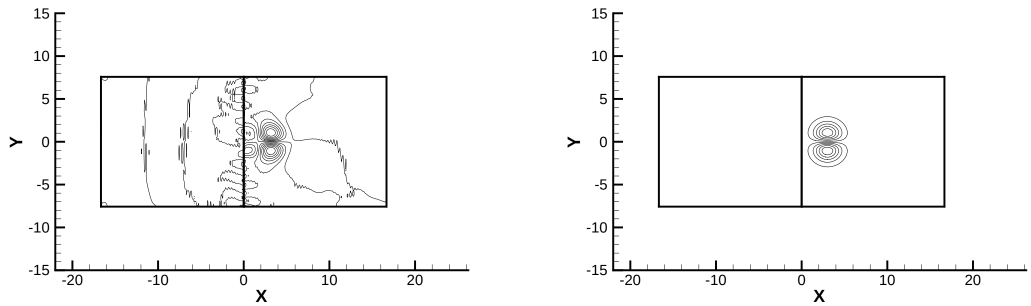
The same contour plots for methods B and C are plotted in Fig 2. As expected, for method C, which is a combination of non-SBP and non-characteristic penalization, we get both numerical errors at the interface and distortion of the vortex. For method B, which is the energy stable combination of SBP operators and characteristic penalties, we get the perfect shape of the vortex with no numerical noise accumulated at the interface.

The conclusion from this study is that only method B, which corresponds to characteristic penalization combined with SBP operators, can be used for stable coupling of two compressible codes. There might, of course, be other stable interface coupling techniques in addition to the methods outlined here. The strength of the SBP and SAT technique (method B) is that we can guarantee that the method is stable and accurate.

3.2. *Compressible-Incompressible coupling*

The same methodology as in the previous section is here used to couple the fourth order accurate and compressible SBP discretization with an unstructured node centered FV incompressible SBP discretization. Since we have a different number of continuous equations in each domain, strictly bounded energy estimates can no longer be obtained. Nevertheless, each side is treated with characteristic (upwind and downwind) interface penalties. We initiate an inviscid Taylor vortex in the compressible domain and let the vortex be swept down by the free stream and enter into the incompressible domain. No artificial dissipation is added, so this is a real test of the stability property of the multi-physics and multi-code coupling. The results are shown in Figure 3. In the incompressible domain, we only penalize the right going (positive) velocities that we obtain from the compressible domain. In the compressible domain, we need to specify all of the primitive variables:

$$u^T = [\rho,\, u,\, v,\, T]\ .$$



(a) Method C.

(b) Method B.

FIGURE 2. Contours of streamwise velocity when the vortex just crossed the interface.

Since the density ρ is constant in the incompressible formulation we penalize against $\rho_\infty = 1$. For temperature, we use $T = T_\infty = 1$. The coupling does not introduce any instabilities (such as the spurious π-mode). We can see that the vortex is slightly distorted (compressed) after entering the incompressible domain. This is probably due to a non-optimal pressure treatment. We believe a more correct coupling will also consider the pressure coupling at the interface. This is something we will analyze further.

4. Conclusions

The results from this study indicate that the SBP and SAT technique is a very robust and efficient method in coupling different flow solvers. Coupling of two compressible solvers can be shown to be stable and accurate using the SAT technique. For coupling of a compressible and an incompressible solver, there are still some improvements to be made. In a coming paper, a new interface coupling that takes into account the pressure terms will be analyzed and tested.

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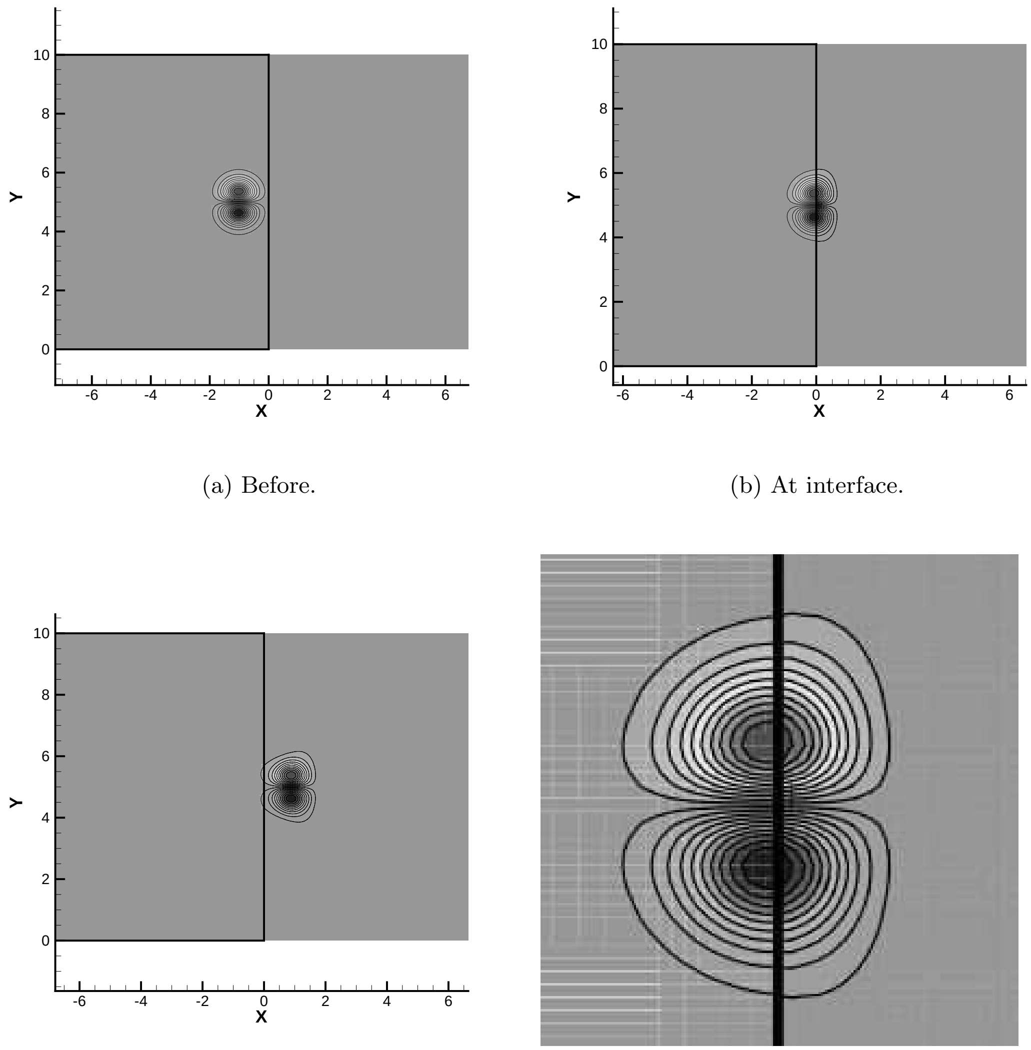
5. Appendix: Difference operators

5.1. *Fourth order accurate SBP operator*

The first and second derivative SBP operators are used in the computations (methods A and B). The interior scheme is a fourth order accurate central scheme with a second



(a) Before.

(b) At interface.

(c) After.

(d) At interface zoom.

FIGURE 3. The velocity contour of the vortex going from compressible to incompressible domain, at three different locations.

order boundary closure. The discrete norm H is defined:

$$H = h \begin{bmatrix} \frac{17}{48} & & & & & \\ & \frac{59}{48} & & & & \\ & & \frac{43}{48} & & & \\ & & & \frac{49}{48} & & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix}.$$

The discrete difference SBP operator approximating $\frac{d}{dx}$ we denote $D = H^{-1}Q$, and is defined:

$$D = \frac{1}{h}\begin{bmatrix} -\frac{24}{17} & \frac{59}{34} & -\frac{4}{17} & -\frac{3}{34} & 0 & 0 & 0 & \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \\ \frac{4}{43} & -\frac{59}{86} & 0 & \frac{59}{86} & -\frac{4}{43} & 0 & 0 & \\ \frac{3}{98} & 0 & -\frac{59}{98} & 0 & \frac{32}{49} & -\frac{4}{49} & 0 & \\ 0 & 0 & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The discrete 4th order accurate SBP operator $D_2 = H^{-1}(-M + BS)$ approximating $\frac{d^2}{dx^2}$ is given by

$$D_2 = \frac{1}{h^2}\begin{bmatrix} 2 & -5 & 4 & -1 & 0 & 0 & 0 & \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & \\ -\frac{4}{43} & \frac{59}{43} & -\frac{110}{43} & \frac{59}{43} & -\frac{4}{43} & 0 & 0 & \\ -\frac{1}{49} & 0 & \frac{59}{49} & -\frac{118}{49} & \frac{64}{49} & -\frac{4}{49} & 0 & \\ 0 & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

and the 3rd order accurate boundary derivative operator BS is given by,

$$S = \frac{1}{h}\begin{bmatrix} -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} & & & & \\ & 1 & & & & & & \\ & & & & \ddots & & & \\ & & & & & & 1 & \\ & & & & -\frac{1}{3} & \frac{3}{2} & -3 & \frac{11}{6} \end{bmatrix}.$$

5.2. *Fourth order accurate non-SBP operator*

The first and second derivative non-SBP operators are used in the computations (methods C and D). The interior scheme is a fourth order accurate central scheme. The first derivative operator has a fourth order accurate boundary closure:

$$\tilde{D} = \frac{1}{12h}\begin{bmatrix} -25 & 48 & -36 & 16 & -3 & 0 & 0 \\ -3 & -10 & 18 & -6 & 1 & 0 & 0 \\ & 1 & -8 & 0 & 8 & -1 & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The second derivative non-SBP operator is given by

$$\tilde{D_2} = \frac{1}{12h^2}\begin{bmatrix} 11 & -20 & 6 & 4 & -1 & 0 & 0 \\ 35 & -104 & 114 & -56 & 11 & 0 & 0 \\ & -1 & 16 & -30 & 16 & -1 & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

and has a third order accurate boundary closure.

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